

ON THE FAILURE OF GORENSTEINNESS FOR THE  
SEQUENCE  $(1, 125, 95, 77, 70, 77, 95, 125, 1)$

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ABSTRACT. In [9], the authors determine an infinite class of non-unimodal Gorenstein sequence, which includes the example

$$\bar{h}_1 = (1, 125, 95, 77, 71, 77, 95, 125, 1).$$

They raise a question whether there is a Gorenstein algebra with Hilbert function

$$\bar{h}_2 = (1, 125, 95, 77, 70, 77, 95, 125, 1),$$

which has remained an open question. In this paper, we prove that there is no Gorenstein algebra with Hilbert function  $\bar{h}_2$ .

## 1. Introduction

Throughout this paper,  $k$  will denote an infinite field, and  $R = k[x_1, \dots, x_r]$  a graded polynomial ring in  $r$  variables. Let  $A = R/I$  be an Artinian graded Gorenstein algebra of socle degree  $e$ . The Hilbert function of  $A$  is a finite symmetric sequence  $\bar{h} = (1, r, h_2, h_3, \dots, h_e)$  and we call it  $h$ -vector of  $A$ . A Gorenstein  $h$ -vector is called unimodal up to the degree  $i + 1$  if  $1 \leq h_1 \leq \dots \leq h_i \leq h_{i+1}$ . A Gorenstein  $h$ -vector is called unimodal if it is unimodal up to the degree  $\lfloor \frac{e}{2} \rfloor$ .

One of important questions is to find the possible Hilbert function of Gorenstein algebra. The complete answer to this question is known for the case  $r \leq 3$  (see [8, 11, 12]). In this case, the  $h$ -vector of Gorenstein sequence is unimodal.

Another important related question is to investigate the unimodality of the Gorenstein  $h$ -vector. In codimension  $\geq 5$ , not all Gorenstein Hilbert functions are unimodal ([1, 3, 5, 11]). It is not known whether

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all Gorenstein algebras of codimension 4 are unimodal. In this context, one may ask if a given non-unimodal sequence is Gorenstein or if there is a good lower bounds for the growth of an Artinian Gorenstein Hilbert function in the first half. In [9], using the classical results of Macaulay, Green and Stanley about binomial expansions and  $h$ -vectors, the authors prove a very important lower bound for  $h_{i+1}$ , for any integer  $1 \leq i \leq \frac{e}{2} - 1$ .

**THEOREM 1.1.** [9, Theorem 2.4] *Suppose that  $\bar{h} = (1, h_1 = r, h_2, \dots, h_{e-2}, h_{e-1}, h_e)$  is the  $h$ -vector of an Artinian Gorenstein algebra over  $R = k[x_1, \dots, x_r]$ . Assume that  $i$  is an integer satisfying  $1 \leq i \leq \frac{e}{2} - 1$ . Then,*

$$h_{i+1} \geq \binom{(h_i)_{(e-i)} - 1}{-1} + \binom{(h_i)_{(e-i)} - (e-2i)}{-(e-2i-1)}.$$

Using this theorem the authors give a very beautiful short proof to the theorem of Stanley that all Gorenstein  $h$ -vectors of codimension  $r \leq 3$  are unimodal. Also this result is used to show that all Gorenstein  $h$ -vectors of codimension 4 and socle degree  $e > \frac{1}{6}(i^2 + 12i + 2)$  are unimodal up to degree  $i + 1$ .

The bound in Theorem 1.1 is often sharp, but one would not expect it to be sharp "too often", since a sharp bound would probably make it easy to decide if non-unimodal Gorenstein Hilbert functions exist. In [9], the authors construct a particular family of Gorenstein algebras with trivial extensions, including the Gorenstein sequence

$$(1, 125, 95, 77, 71, 77, 95, 125, 1),$$

and raise a question whether there is a Gorenstein algebra with Hilbert function

$$\bar{h}_2 = (1, 125, 95, 77, 70, 77, 95, 125, 1),$$

which has remained an open question. In this paper, we prove that there is no Gorenstein algebra with Hilbert function  $\bar{h}_2$ .

## 2. Gorenstein Hilbert functions

Recall that if  $n$  and  $i$  are positive integers, then  $n$  can be written uniquely in the form

$$n_{(i)} = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \dots + \binom{n_j}{j},$$

where  $n_i > n_{i-1} > \dots > n_j \geq j \geq 1$  (see Lemma 4.2.6, [4]).

Following [2], we define, for any integers  $a$  and  $b$ ,

$$\binom{n_{(i)}}{b}^a = \binom{n_i + a}{i + b} + \binom{n_{i-1} + a}{i - 1 + b} + \cdots + \binom{n_j + a}{j + b}$$

where  $\binom{m}{n} = 0$  for either  $m < n$  or  $n < 0$ .

**THEOREM 2.1** ([6], Chapter 5 in [7]). *Let  $L$  be a general linear form in  $R$  and we denote by  $h_d$  the degree  $d$  entry of the Hilbert function of  $R/I$  and  $\ell_d$  the degree  $d$  entry of the Hilbert function of  $R/(I, L)$ . Then, we have the following inequalities.*

- (a) Macaulay’s Theorem:  $h_{d+1} \leq ((h_d)_{(d)})_1^1$ .
- (b) Green’s Hyperplane Restriction Theorem:  $\ell_d \leq ((h_d)_{(d)})_0^{-1}$ .

**LEMMA 2.2** ([11]). *Let  $A = R/I$  be an Artinian Gorenstein algebra, and let  $L \notin I$  be a linear form of  $R$ . Then the  $h$ -vector of  $A$  can be written as*

$$h := (h_0, h_1, \dots, h_s) = (1, b_1 + \ell_1, \dots, b_{s-1} + \ell_{s-1}, b_s = 1)$$

where  $b = (b_1 = 1, b_2, \dots, b_{s-1}, b_s = 1)$  is the  $h$ -vector of  $R/(I : L)$  (with the indices shifted by 1), which is a Gorenstein algebra, and

$$\ell = (\ell_0, \ell_1, \dots, \ell_{s-1}) \quad \text{with } \ell_0 = 1$$

is the  $h$ -vector of  $R/(I, L)$ .

Let  $S$  be an Artinian graded  $k$ -algebra with socle degree  $e$ . Then  $E = \text{Hom}(S, k)$  is the injective envelope of  $k$ , regarded as the residue class field of  $S$ .  $E$  has the structure of  $S$ -module in the usual way,  $(x\phi)(y) = \phi(xy)$ , where  $x \in S$  and  $\phi \in E$ . Let  $R = S \times E$ , endowed with component wise addition and the multiplication  $(x, \phi) \cdot (y, \psi) = (xy, x\psi + y\phi)$ . With these notations, we have

**LEMMA 2.3** (Trivial Extension). *A graded algebra  $R$  is a 0-dimensional Gorenstein  $k$ -algebra. Moreover, its Hilbert function satisfies*

$$H(R, d) = H(S, d) + H(S, e + 1 - d), \quad (0 \leq d \leq e + 1).$$

This is a way to construct Gorenstein algebra and we call  $R$  trivial extension algebra induced by  $S$ .

The set of all Artinian Gorenstein algebras of codimension  $\leq r$  and socle degree  $e$  arises, via Macaulay’s inverse systems (or divided powers in characteristic  $p$ ), by looking at the annihilators of forms  $F \in R_e$ . So the set of all such algebras is parametrized by the projective space  $\mathbb{P}(R_e)$ . Macaulay’s inverse system is another way of constructing Gorenstein

algebras. Using it, we have the following result. [13, Proposition 8] Suppose that  $(1, h_1, h_2, \dots, h_{e-1}, 1)$  is a Gorenstein  $h$ -vector. Then also

$$(1, h_1 + 1, h_2 + 1, \dots, h_{e-1} + 1, 1)$$

is always a Gorenstein  $h$ -vector.

**3. The sequence  $(1, 125, 95, 77, 70, 77, 95, 125, 1)$  is not a Gorenstein sequence.**

In [9], Migliore-Nagel-Zanello constructed a Gorenstein sequence

$$(1, 125, 95, 77, 71, 77, 95, 125, 1),$$

Indeed, let  $S = k[x_1, x_2, x_3, x_4]/(x_1, x_2, x_3, x_4)^8$  and  $E = \text{Hom}_k(S, k)$ . Note that  $S$  has the Hilbert function

|           |   |   |    |    |    |    |    |     |   |
|-----------|---|---|----|----|----|----|----|-----|---|
| $d$       | 0 | 1 | 2  | 3  | 4  | 5  | 6  | 7   | 8 |
| $H(S, d)$ | 1 | 4 | 10 | 20 | 35 | 56 | 84 | 120 | 0 |

Then  $E$  has the structure of an  $S$ -module in the usual way, i.e.,

$$(f\varphi)g = \varphi(fg),$$

where  $\varphi \in E$  and  $f, g \in R$ . Let  $T = R \times E$ , endowed with component-wise addition and the multiplication

$$(x, \varphi) \cdot (y, \psi) = (xy, x\psi + y\varphi).$$

Then  $T$  is a 0-dimensional Gorenstein  $k$ -algebra (see [11, Example 4.3]) with Hilbert function  $H(T, d) = H(S, d) + H(S, 8 - d)$  and thus, we have

$$(3.1) \quad \begin{array}{c|cccccccccc} d & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline H(T, d) & 1 & 124 & 94 & 76 & 70 & 76 & 94 & 124 & 1 \end{array}$$

Since there is a Gorenstein algebra with Hilbert function (3.1), it follows from [13, Proposition 8], the following  $h$ -vector

$$(1, 125, 95, 77, 71, 77, 95, 125, 1)$$

is a Gorenstein sequence. Note that this sequence satisfies the equality in Theorem 1.1 in each degree  $d \leq 3$ . For the case  $d = 4$ , we have

$$h_4 = 71 > ((h_3)_{(5)})_{-1}^{-1} + ((h_3)_{(5)})_{-1}^{-2} = 70.$$

In [9], the authors raise a question whether there is a Gorenstein algebra with Hilbert function

$$\bar{h} = (1, 125, 95, 77, 70, 77, 95, 125, 1),$$

which has remained an open question. Our main result shows that there is no Gorenstein algebra with Hilbert function  $\bar{h}$ .

**THEOREM 3.1.**  $\bar{h} = (1, 125, 95, 77, 70, 77, 95, 125, 1)$  is not a Gorenstein sequence.

*Proof.* Suppose that  $\bar{h} = (1, 125, 95, 77, 70, 77, 95, 125, 1)$  is a Gorenstein sequence. Then there is an Artinian Gorenstein algebra  $A = R/I$  with the Hilbert function  $\bar{h}$ . and let  $L \notin I$  be a linear form of  $R$ . Then the  $h$ -vector of  $A$  can be written as

$$\bar{h} = (h_0, h_1, \dots, h_s) = (1, b_1 + \ell_1, \dots, b_{s-1} + \ell_{s-1}, b_s = 1)$$

where  $\bar{b} = (b_1 = 1, b_2, \dots, b_{s-1}, b_s = 1)$  is the  $h$ -vector of  $R/(I : L)$  (with the indices shifted by 1), which is a Gorenstein algebra, and

$$\bar{\ell} = (\ell_0, \ell_1, \ell_2, \dots, \ell_{s-1}) \quad \text{with } \ell_0 = 1, \ell_1 = 124$$

is the  $h$ -vector of  $R/(I, L)$ . So we have the following table:

|          |   |     |          |          |          |          |          |          |   |
|----------|---|-----|----------|----------|----------|----------|----------|----------|---|
| $d$      | 0 | 1   | 2        | 3        | 4        | 5        | 6        | 7        | 8 |
| $h_d$    | 1 | 125 | 95       | 77       | 70       | 77       | 95       | 125      | 1 |
| $\ell_d$ | 1 | 124 | $\ell_2$ | $\ell_3$ | $\ell_4$ | $\ell_5$ | $\ell_6$ | $\ell_7$ | 0 |
| $b_d$    | 0 | 1   | $b_2$    | $b_3$    | $b_4$    | $b_5$    | $b_6$    | $b_7$    | 1 |

Note that Green's Restriction Theorem implies  $\ell_i \leq (h_i)_0^{-1}$  for each  $i \geq 0$ . From the following table

|                |   |     |    |    |    |    |    |     |   |
|----------------|---|-----|----|----|----|----|----|-----|---|
| $d$            | 0 | 1   | 2  | 3  | 4  | 5  | 6  | 7   | 8 |
| $h_d$          | 1 | 125 | 95 | 77 | 70 | 77 | 95 | 125 | 1 |
| $(h_d)_0^{-1}$ | 1 | 124 | 81 | 50 | 35 | 27 | 30 | 36  | 0 |

we have  $(1, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \ell_7, \ell_8) <_{\text{Lex}} (1, 124, 81, 50, 35, 27, 30, 36)$ .

Now let  $\ell_5 = 27 - i$  for some  $i \geq 0$  and  $b_5 = h_5 - \ell_5 = 50 + i$ . Since  $\bar{b}$  is a Gorenstein sequence,  $\bar{b}$  should be symmetric. So we have  $b_4 = b_5 = 50 + i$  and thus  $\ell_4 = h_4 - b_4 = 20 - i$ .

For such  $i$  with  $0 \leq i \leq 19$ , consider the following table:

|                      |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |     |
|----------------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|-----|
| $i$                  | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | ... |
| $\ell_4 = 20 - i$    | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | 10 | 9  | 8  | 7  | 6  | 5  | 4  | ... |
| $(\ell_4)_{+1}^{+1}$ | 27 | 26 | 24 | 23 | 22 | 21 | 18 | 16 | 15 | 13 | 12 | 11 | 9  | 8  | 7  | 6  | 4  | ... |
| $\ell_5 = 27 - i$    | 27 | 26 | 25 | 24 | 23 | 22 | 21 | 20 | 19 | 18 | 17 | 16 | 15 | 14 | 13 | 12 | 11 | ... |

By Macaulay's Theorem, we see that the following inequality has to be satisfied,

$$\ell_5 \leq (\ell_4)_{+1}^{+1}$$

and it happens if and only if  $i = 0, 1$ . So we have  $\ell_4 = 19$  or  $20$ .

On the other hand, if  $\ell_6 = 30 - j$  then  $b_3 = b_6 = h_6 - \ell_6 = 65 + j$  and thus  $\ell_3 = h_3 - b_3 = 12 - j \leq 12$ . Since the numerical function  $(-)^{+1}_{+1}$  is a strictly increasing function, we have

$$19 \leq \ell_4 \leq (\ell_3)^{+1}_{+1} \leq (12_{(3)})^{+1}_{+1} = 17,$$

which is impossible. Therefore, we conclude that the sequence

$$(1, 125, 95, 77, 70, 77, 95, 125, 1)$$

is not a Gorenstein sequence.  $\square$

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